A Mathematical Model of Stock Price and Some Related Analysis

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Abstract

A mathematical model is used to characterize the price of a stock over time. By utilizing the model, we can estimate the average and variation of the stock price, and hence predict the trend of the future stock price.

Model: $X_t = X_0 + \mu t + \sigma W_t$

$W_{\Delta t}, W_{2\Delta t} - W_{\Delta t}, W_{3\Delta t} - W_{2\Delta t}, \dots, W_{(k+1)\Delta t} - W_{k\Delta t}$

(They are independent)

$$X_{(k+1)\Delta t} = x_0 + \mu(k+1)\Delta t + \sigma W_{(K+1)\Delta t}$$

$$= \underbrace{x_0 + \mu k\Delta t + \sigma W_{k\Delta t}}_{X_{k\Delta t}} + \mu\Delta t + \underbrace{\sigma(W_{(k+1)\Delta t} - W_{k\Delta t})}_{\sqrt{\Delta t}\cdot Z_{k+1}}$$

$$Y_1 = X_{\Delta t} - X_0 = \mu\Delta t + \sigma\sqrt{\Delta t}Z_1$$

$$Y_2 = X_{2\Delta t} - X_{\Delta t} = \mu\Delta t + \sigma\sqrt{\Delta t}Z_2$$

$$\vdots$$

$$Y_i = \mu\Delta t + \sigma\sqrt{\Delta t}Z_i$$

$Y_1, Y_2, Y_3, \dots, Y_i$ are all independent

$$\mathrm{E}(Y_i) = E(\mu\Delta t + \sigma\sqrt{\Delta t}Z_i) = \mu\Delta t$$

$$\mathrm{Var}(Y_i) = E(Y_i - E(Y))^2 = E(\sigma\sqrt{\Delta t}Z_i)^2 = \sigma^2\Delta t E(Z_i^{\,2}) = \Delta t\sigma^2$$

Then, $\hat{\mu}$ and $\hat{\sigma}$ can be estimated by using

$$\hat{\mu} = \frac{\sum_{i=0}^{\infty} Y_i}{T}, \quad \hat{\sigma} = \sqrt{\frac{s^2}{\Delta t}}$$

Stock price:

$$\boldsymbol{S_t = S_0 \cdot exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}}$$

$$\ln S_t = (\ln S_t) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t$$

$\Rightarrow \ln S_t = X_t$

Prediction:

(This is for the predicted stock price in the future)

$$\boldsymbol{S_t = \underbrace{S_{10}}_{\text{The stock price at time 10}} \cdot \exp\{(\hat{\mu} - \tfrac{1}{2}\hat{\sigma}^2)(t-10)\}} \qquad t > 10$$

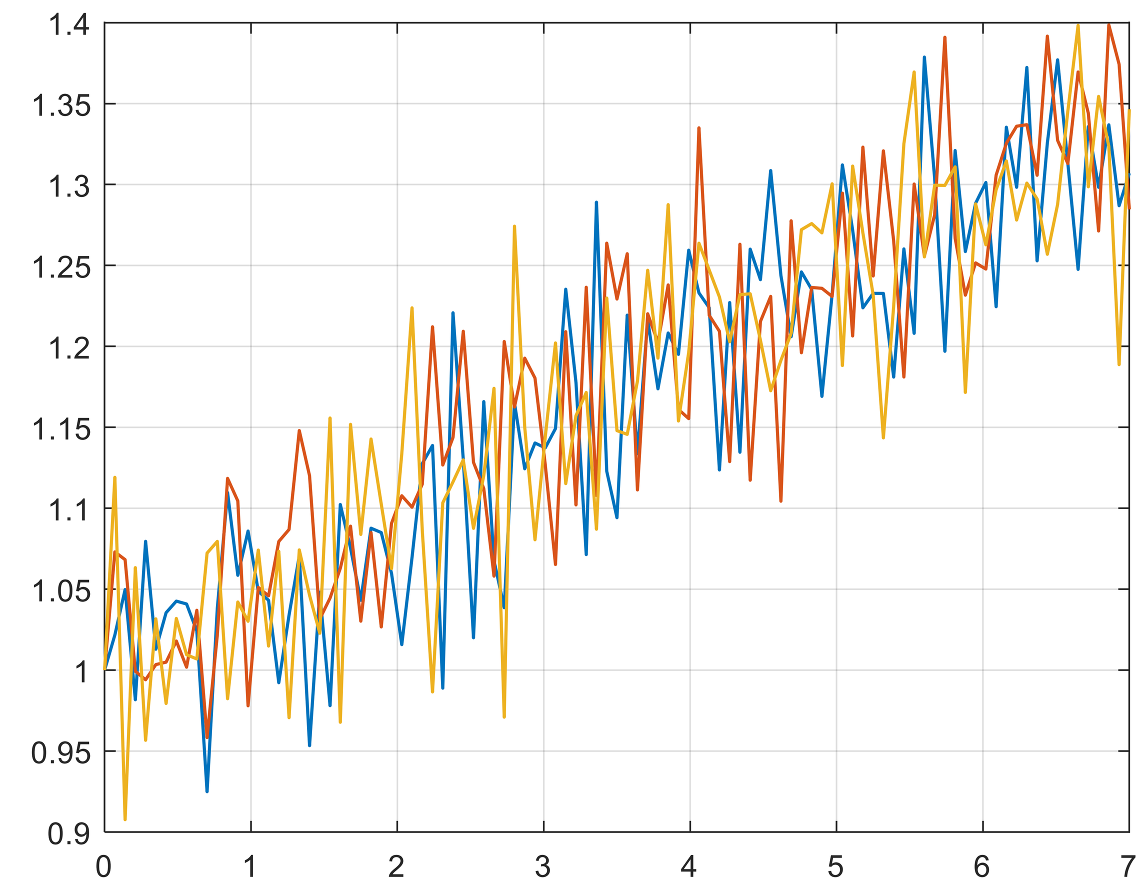

Figure 1: This plot shows several simulated path of X_t with $x_0 = 1$, $\boldsymbol{\mu = 5\%, \sigma = 20\%}$

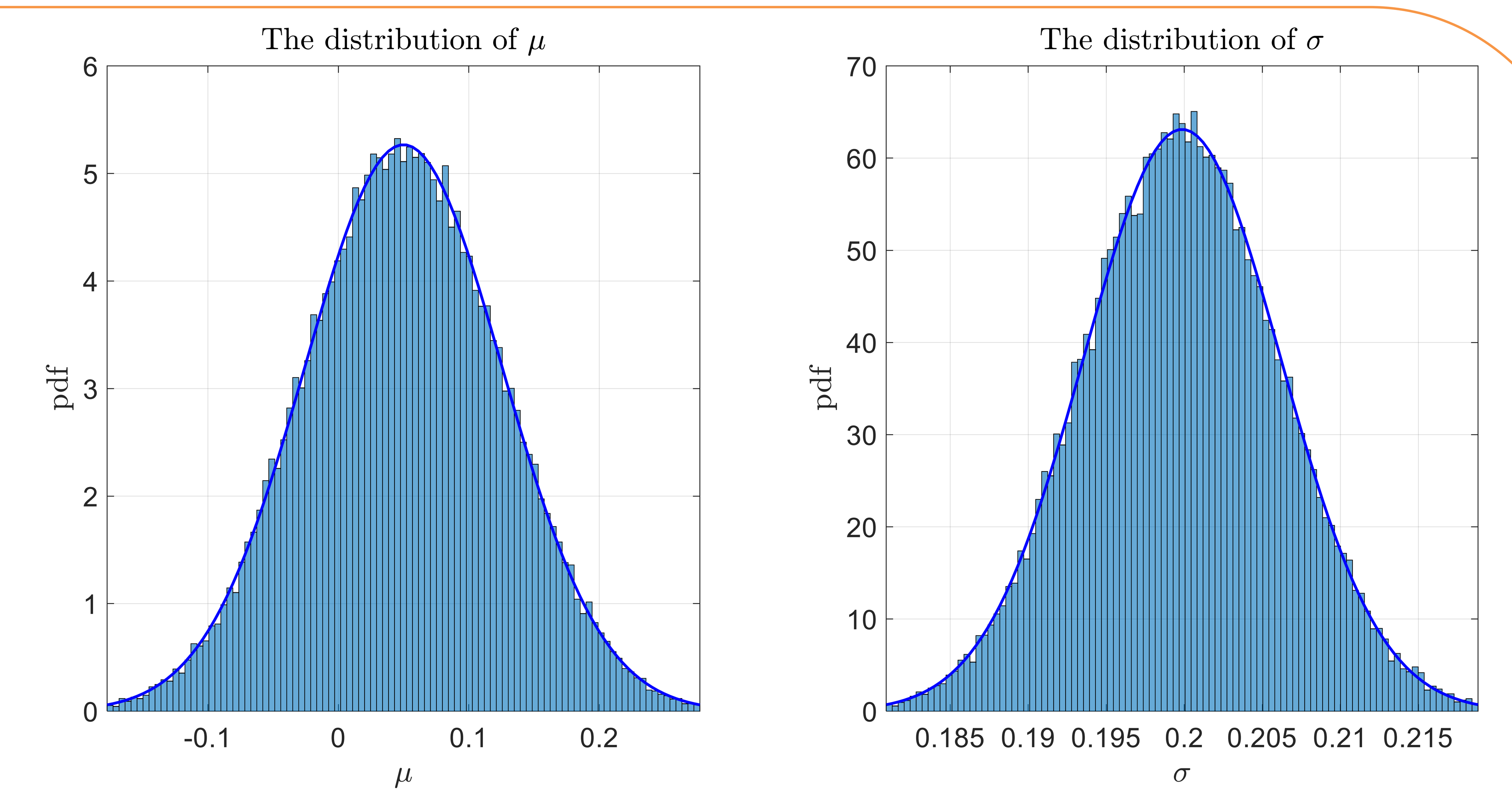

Figure 2: The left bar plot for the estimator $\hat{\mu}$ from 50000 simulated paths shows a normality shape. Similar to the estimator $\hat{\sigma}$ shown on the right bar plot.

The mean of μ is 4.991%. **$\boldsymbol{\delta}$ is 0.09%** (δ is the estimation error**)**
The 90% CI of μ is (-7.465%, 17.447%). The 95% CI of μ is (-9.852%, 19.834%)
The mean of σ is 19.999%. **$\boldsymbol{\delta}$ is 0.001%** (δ is the estimation error**)**
The 90% CI of σ is [18.946%,21.0265%]. The 95% CI of σ is [18.747%,21.225%]

The most popular model for a stock price is geometric Brownian motion (GBM)

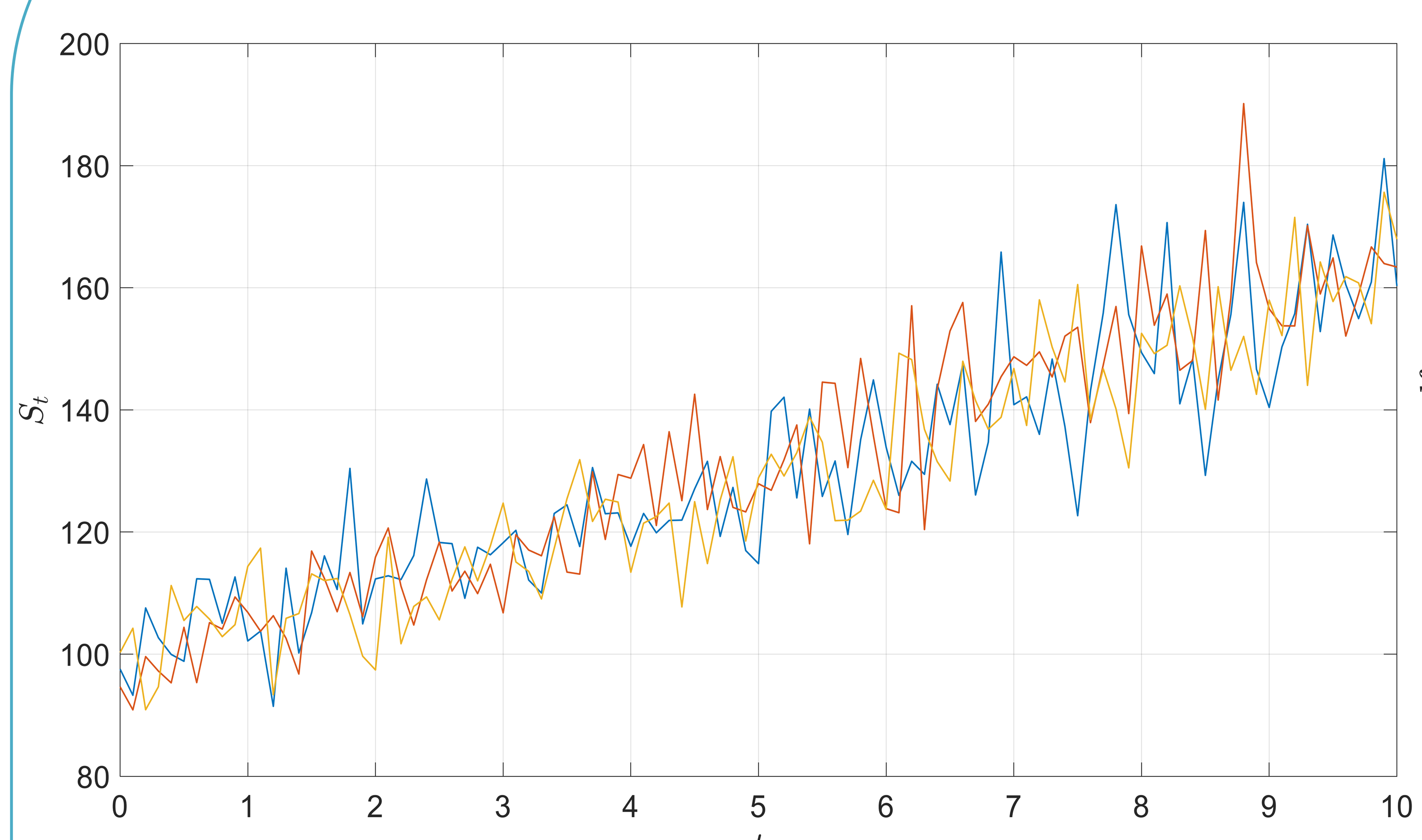

Figure 3: This plot shows several simulated path of S_t with $S_0 = 100$, $\boldsymbol{\mu = 7\%, \sigma = 20\%}$

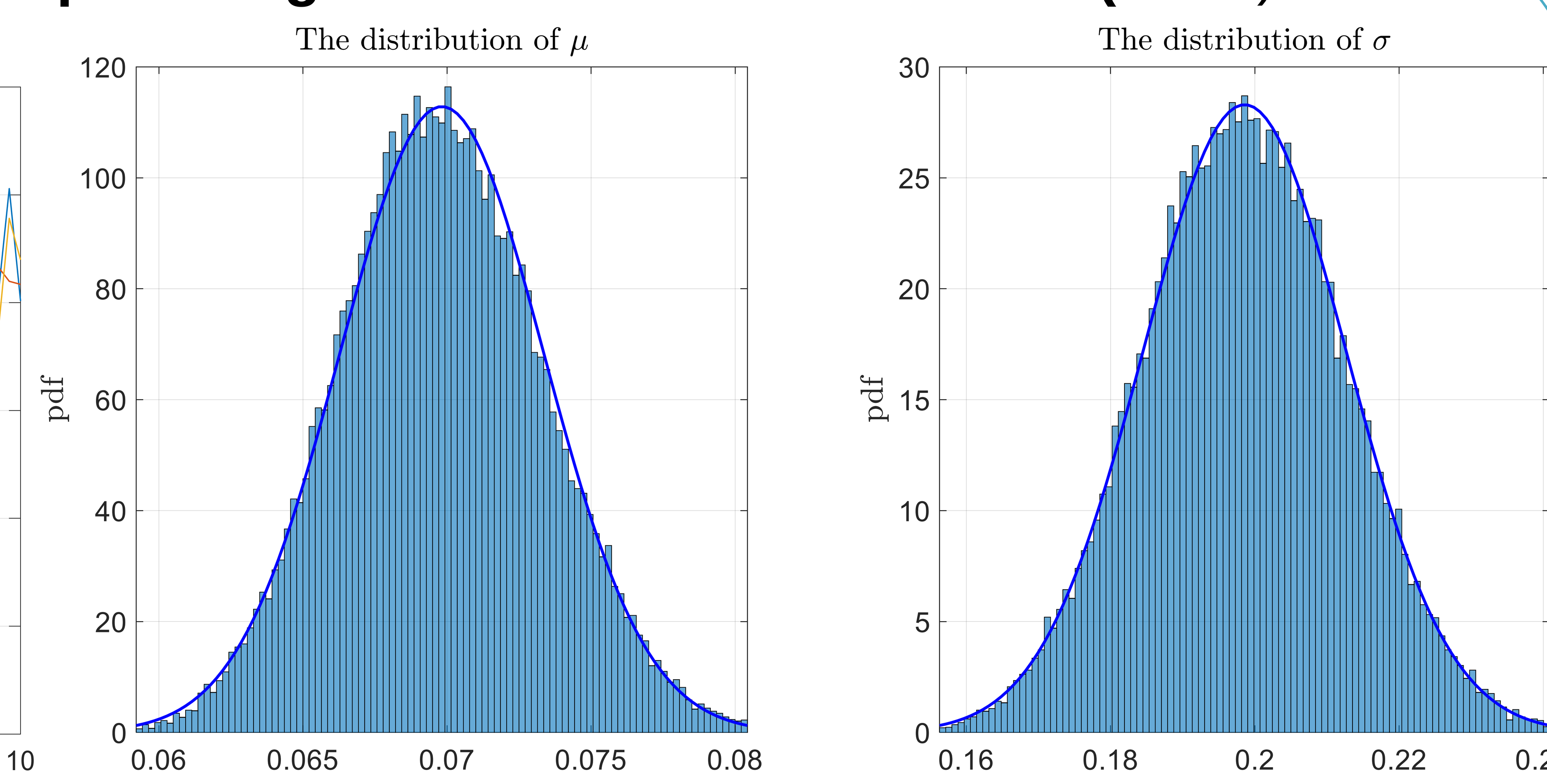

Figure 4: The left bar plot for the estimator $\hat{\mu}$ from 50000 simulated paths shows a normality shape. Similar to the estimator $\hat{\sigma}$ shown on the right bar plot.

The mean of μ is 6.982%. **$\boldsymbol{\delta}$ is 0.018%** (δ is the estimation error)
The 90% CI of μ is (6.400%,7.563%). The 95% CI of μ is (6.289%,7.675%)
The mean of σ is 19.855%. **$\boldsymbol{\delta}$ is 0.145%** (δ is the estimation error)
The 90% CI of σ is [17.539%,22.177%]. The 95% CI of σ is [17.095%,22.622%]